Vacuum stability conditions of the general two-Higgs-doublet potential

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Abstract

In this paper, we present the novel analytical expressions for the bounded-from-below or the vacuum stability conditions of scalar potential for a general two-Higgs-doublet model by using the concepts of co-positivity and the gauge orbit spaces. More precisely, several sufficient conditions and necessary conditions are established for the vacuum stability of the general 2HDM potential, respectively. We also give an equivalent condition of the vacuum stability of the general 2HDM potential in theory, and then, apply it to derive the analytical necessary conditions of the general 2HDM potential. Meanwhile, the semi-positive definiteness is proved for a class of 4th-order 2-dimensional complex tensor.

Keywords: Co-positivity, Complex tensors, CP violation, 2HDM

1 Introduction

In 1973, the fist two-Higgs-doublet model (for short, 2HDM) is presented by Lee [1, 2]. Subsequently, Weinberg [3] proposed a general multi-Higgs potential model. Since then, the stability of the scalar Higgs potential is an important problem with the Standard Model (for short, SM) at high-energies. The bounded-from-below (for short, BFB) or the vacuum stability of SM is very noticeable in particle physics community. One of the simplest extensions of the SM Higgs sector is the 2HDM [1, 4]. It is well-known that the most general Higgs potential for a 2HDM with Higgs doublets Φ_1 and Φ_2 can be written [5–8]

$$\begin{aligned} V_H(\Phi_1,\Phi_2) =& \mu_{11}\Phi_1^*\Phi_1 + \mu_{22}\Phi_2^*\Phi_2 - (\mu_{12}\Phi_1^*\Phi_2 + \mu_{12}^*\Phi_2^*\Phi_1) \\ &+ \lambda_1(\Phi_1^*\Phi_1)^2 + \lambda_2(\Phi_2^*\Phi_2)^2 \\ &+ \lambda_3(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2) + \lambda_4(\Phi_1^*\Phi_2)(\Phi_2^*\Phi_1) \\ &+ \frac{\lambda_5}{2}(\Phi_1^*\Phi_2)^2 + \frac{\lambda_5^*}{2}(\Phi_2^*\Phi_1)^2 \\ &+ (\Phi_1^*\Phi_1)(\lambda_6\Phi_1^*\Phi_2 + \lambda_6^*\Phi_2^*\Phi_1) \\ &+ (\Phi_2^*\Phi_2)(\lambda_7\Phi_1^*\Phi_2 + \lambda_7^*\Phi_2^*\Phi_1), \end{aligned} \tag{1}$$

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where Φ^* is Hermitian conjugate of Φ. The parameters μ_{11}, μ_{22} and λ_i $(i = 1, 2, 3, 4)$ are real, μ_{12} and λ_i $(i = 5, 6, 7)$ are complex.

The Higgs potential of such a 2HDM (1) may be said [7, 9]

$$V_H(\Phi_1, \Phi_2) = \sum_{a,b=1}^{2} \mu_{ab} \Phi_a^* \Phi_b + \sum_{i,j,k,l=1}^{2} t_{ijkl} (\Phi_i^* \Phi_j)(\Phi_k^* \Phi_l), \tag{2}$$

where, by definition,

$$t_{ijkl} = t_{klij},\ t_{ijkl} = t_{jilk}^*,\ \mu_{ab} = \mu_{ba}^*. \tag{3}$$

The quartic part,

$$V_4(\Phi_1, \Phi_2) = \sum_{i,j,k,l=1}^{2} t_{ijkl} (\Phi_i^* \Phi_j)(\Phi_k^* \Phi_l), \tag{4}$$

gives a 4th-order 2-dimensional complex tensor $\mathcal{T} = (t_{ijkl})$:

$$\begin{aligned}
t_{1111} =& \lambda_1,\ t_{2222} = \lambda_2,\\
t_{1122} =& t_{2211} = \frac{1}{2}\lambda_3,\ t_{1221} = t_{2112} = \frac{1}{2}\lambda_4,\\
t_{1212} =& \frac{1}{2}\lambda_5,\ t_{2121} = \frac{1}{2}\lambda_5^*,\\
t_{1112} =& t_{1211} = \frac{1}{2}\lambda_6,\ t_{1121} = \ t_{2111} = \frac{1}{2}\lambda_6^*,\\
t_{1222} =& t_{2212} = \frac{1}{2}\lambda_7,\ t_{2122} = \ t_{2221} = \frac{1}{2}\lambda_7^*.
\end{aligned} \tag{5}$$

The stability of the 2HDM potential requires that there is no direction in field space along which the potential tends to minus infinity, i.e., it is the BFB. In general, the quartic part of the scalar potential, V_4, is non-negative for arbitrarily large values of the component fields, but the quadratic part of the scalar potential, V_2, can take negative values for at least some values of the fields [7]. Considering only the quartic part V_4, the condition for stability (the BFB) of the scalar potential in the 2HDM is equivalent to the co-positivity or semi-positive definiteness of the tensor $\mathcal{T} = (t_{ijkl})$ given by the Higgs quartic coupling λ_i, i.e. $V_4(\Phi_1, \Phi_2) \geq 0$. When $\lambda_6 = \lambda_7 = 0$, the vacuum stability conditions of 2HDM potential [4, 10, 11, 13–15] are the following:

$$\lambda_1 > 0, \lambda_2 > 0, \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0, \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0.$$

In 2011, Battye et al [16] employed the method using Sylvester's criterion and Lagrange multipliers for the general CP-violating 2HDM. In 2016, Kannike [17–19] presented the vacuum stability conditions of the scalar potential of two Higgs doublets in the 2HDM with explicit CP conservation. Chauhan [20] derived analytical necessary and sufficient conditions for the vacuum stability of the left-right symmetric model, and gave the sufficient conditions for successful symmetry breaking. Recently, Song [21] showed the analytical sufficient and necessary conditions of the co-positivity of the tensor $\mathcal{T} = (t_{ijkl})$ with the real nembers λ_i $(i = 5, 6, 7)$,

and moreover, the vacuum stability conditions of scalar potential for the 2HMD with explicit CP conservation was obtained. Bahl et.al. [22] gave the analytical sufficient conditions of the vacuum stability for the 2HMD potential with CP conservation and CP violation, respectively, where the vacuum stability condition for the 2HMD potential with CP violation depends on the Lagrange multiplier ζ. For more details about the BFB or the vacuum stability conditions of the 2HDM potential, see Refs. [11,12,23–26] for 2HDM with CP conservation; Refs. [8,24] for the most general 2HDM; Refs. [7,22,24] for 2HDM with CP conservation and CP violation; Ref. [27] for 2HDM handled numerically and others references that are no cited here.

In this paper, we provide three new analytical sufficient conditions for the bounded-from-below or the vacuum stability of scalar potential for a general 2HDM by using the co-positivity of 4th-order 2-dimensional symmetric real tensor, which is different from the ones of Bahl et.al. [22]. For example,

$$\lambda_1 = \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = 1, \lambda_5 = -1 - 3i, \lambda_6 = \lambda_7 = -2i.$$

Obviously, $\mathbf{Re}\lambda_6 = \mathbf{Re}\lambda_7 = 0, \lambda_3 + 2\sqrt{\lambda_1\lambda_2} = -1 + 2 > 0,$

$$\lambda_3 + \lambda_4 - |\mathbf{Re}\lambda_5| + 2\sqrt{\lambda_1\lambda_2} = -1 + 1 - 1 + 2 > 0,$$
$$\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7 > 0, \ -|\mathbf{Im}\lambda_5| + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} = -3 + 4 > 0.$$

That is, these parameters meet the conditions (**I**) and (**III**) , which means $V_4(\Phi_1, \Phi_2) \geq 0$. It is obvious that the conditions ($\mathbf{IV'}$) and (**III**) are fulfiled also. Here, the conditions (**I**), (**III**) and ($\mathbf{IV'}$) will be derived in Section 3. However, they can't satisfy the condition Eq. (5.20) of Bahl et.al. [22], i.e.

$$3\sqrt{\lambda_1\lambda_2} - (\lambda_3 + |\lambda_4| + |\lambda_5|) = 3 - (-1 + 1 + \sqrt{10}) < 0,$$
$$\sqrt{\lambda_1\lambda_2} + \lambda_3 - (|\lambda_4| + |\lambda_5| + 4\left|\lambda_6\sqrt[4]{\frac{\lambda_2}{\lambda_1}}\right|) = 1 - 1 - (1 + \sqrt{10} + 8) < 0,$$
$$\sqrt{\lambda_1\lambda_2} + \lambda_3 - (|\lambda_4| + |\lambda_5| + 4\left|\lambda_7\sqrt[4]{\frac{\lambda_1}{\lambda_2}}\right|) = 1 - 1 - (1 + \sqrt{10} + 8) < 0.$$

A sufficient and necessary condition of the vacuum stability of the general 2HDM potential is given in theory, which contains the vacuum stability condition of the general 2HDM potential with $\mathbb{Z}_2$ symmetry as a special case. Then, we apply this conclusion to derive the analytical necessary conditions of the vacuum stability of a general 2HDM scalar potential. Meanwhile, the analytical sufficient conditions and necessary conditions are obtained for the semi-positive definiteness of a class of 4th-order 2-dimensional complex tensor.

2 Co-positivity criteria

The co-positivity of a matrix $\mathbf{M} = (\mu_{ij})$ has been applied to test the vacuum stability of the 2HDM in Refs. [17–20]. It is kown-well that a

2×2 symmetric real matrix $\mathbf{M} = (\mu_{ij})$ is co-positive, i.e., for all non-negative vectors $\mathbf{x} = (x_1, x_2)^\top \in \mathbb{R}^2$, the quadratic form

$$\mathbf{x}^\top \mathbf{M}\mathbf{x} = \mu_{11}x_1^2 + 2\mu_{12}x_1x_2 + \mu_{22}x_2^2 \geq 0,$$

if and only if [28–30]

$$\mu_{11} \geq 0, \mu_{22} \geq 0 \text{ and } \mu_{12} + \sqrt{\mu_{11}\mu_{22}} \geq 0. \tag{6}$$

The co-positivity of a symmetric real tensor has been used to the SM in literature to obtain vacuum stability conditions in Refs. [17, 21, 31–34]. A 4th-order n-dimensional symmetric real tensor $\mathcal{T} = (t_{ijkl})$ is co-positive [35–40] if for all non-negative vectors $\mathbf{x} = (x_1, x_2, \cdots x_n)^\top \in \mathbb{R}^n$, the quartic form

$$\mathcal{T}\mathbf{x}^4 = \sum_{i,j,k,l=1}^{n} t_{ijkl}x_ix_jx_kx_l \geq 0.$$

Let $t_{1111} = \alpha_0 > 0$ and $t_{2222} = \alpha_4 > 0$. Recently, Song and Li [32] presented the analytical expressions of co-positivity of a 4th-order 2-dimensional symmetric $\mathcal{T}$ with the help of the update version ([37]) of Ulrich and Watson's result [41]. Let $\mathcal{T} = (t_{ijkl})$ be a 4th-order 2-dimensional symmetric real tensor with its entires

$$t_{1111} = \alpha_0, t_{2222} = \alpha_4, t_{1112} = \frac{1}{4}\alpha_1, t_{1122} = \frac{1}{6}\alpha_2, t_{1222} = \frac{1}{4}\alpha_3.$$

Then the quartic form

$$\mathcal{T}\mathbf{x}^4 = \alpha_0x_1^4 + \alpha_1x_1^3x_2 + \alpha_2x_1^2x_2^2 + \alpha_3x_1x_2^3 + \alpha_4x_2^4 \geq 0 \tag{7}$$

for all $x_1 \geq 0, x_2 \geq 0$ if and only if

$$\begin{aligned}
&(1)\Delta \leq 0, \alpha_1\sqrt{\alpha_4} + \alpha_3\sqrt{\alpha_0} > 0;\\
&(2)\alpha_1 \geq 0, \alpha_3 \geq 0, 2\sqrt{\alpha_0\alpha_4} + \alpha_2 \geq 0;\\
&(3)\Delta \geq 0, |\alpha_1\sqrt{\alpha_4} - \alpha_3\sqrt{\alpha_0}| \leq 4\sqrt{\alpha_0\alpha_2\alpha_4 + 2\alpha_0\alpha_4\sqrt{\alpha_0\alpha_4}},\\
&\quad (i) - 2\sqrt{\alpha_0\alpha_4} \leq \alpha_2 \leq 6\sqrt{\alpha_0\alpha_4},\\
&\quad (ii)\alpha_2 > 6\sqrt{\alpha_0\alpha_4}\\
&\quad \alpha_1\sqrt{\alpha_4} + \alpha_3\sqrt{\alpha_0} \geq -4\sqrt{\alpha_0\alpha_2\alpha_4 - 2\alpha_0\alpha_4\sqrt{\alpha_0\alpha_4}},
\end{aligned}$$

where $\Delta = 4(12\alpha_0\alpha_4 - 3\alpha_1\alpha_3 + \alpha_2^2)^3 - (72\alpha_0\alpha_2\alpha_4 + 9\alpha_1\alpha_2\alpha_3 - 2\alpha_2^3 - 27\alpha_0\alpha_3^2 - 27\alpha_1^2\alpha_4)^2$.

Song and Qi [33, Theorem 3.7] gave a stronger sufficient condition for the co-positivity of a symmetric real tensor $\mathcal{T} = (t_{ijkl})$. That is, $\mathcal{T}\mathbf{x}^4 \geq 0$ for all $x_1 \geq 0, x_2 \geq 0$ if

$$\begin{aligned}
&\beta = \alpha_1 + 4\sqrt[4]{\alpha_0^3\alpha_4} \geq 0, \gamma = \alpha_3 + 4\sqrt[4]{\alpha_0\alpha_4^3} \geq 0,\\
&\alpha_2 - 6\sqrt{\alpha_0\alpha_4} + 2\sqrt{\beta\gamma} \geq 0.
\end{aligned} \tag{8}$$

Song [21] obtained an analytical sufficient and necessary condition for the co-positivity of a symmetric real tensor $\mathcal{T}(\rho,\theta) = (t_{ijkl}(\rho,\theta))$ with two parameters $\rho \in [0,1]$ and $\theta \in [0,2\pi]$,

$$\begin{aligned} t_{1111} =&\Lambda_1, t_{2222} = \Lambda_2, \\ t_{1122} =&\frac{1}{6}(\Lambda_3 + \Lambda_4\rho^2 + \Lambda_5\rho^2\cos 2\theta), \\ t_{1112} =&\frac{1}{2}\Lambda_6\rho\cos\theta,\ t_{1222} = \frac{1}{2}\Lambda_7\rho\cos\theta. \end{aligned} \tag{9}$$

That is, $\Lambda_1 > 0$, $\Lambda_2 > 0$, the quartic form

$$\begin{aligned} \mathcal{T}(\rho,\theta)\mathbf{x}^4 =&\Lambda_1 x_1^4 + \Lambda_2 x_2^4 + (\Lambda_3 + \Lambda_4\rho^2 + \Lambda_5\rho^2\cos 2\theta)x_1^2x_2^2 \\ &+ 2(\rho\Lambda_6\cos\theta)x_1^3x_2 + 2(\rho\Lambda_7\cos\theta)x_1x_2^3 \geq 0, \end{aligned} \tag{10}$$

for all $x_1 \geq 0, x_2 \geq 0$ if and only if

(**a**) $\Lambda_6 = \Lambda_7 = 0, \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0, \Lambda_3 + \Lambda_4 - |\Lambda_5| + 2\sqrt{\Lambda_1\Lambda_2} \geq 0;$

(**b**) $\Delta \geq 0, \Lambda_3 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0, \Lambda_3 + \Lambda_4 - \Lambda_5 + 2\sqrt{\Lambda_1\Lambda_2} \geq 0,$

$|\Lambda_6\sqrt{\Lambda_2} - \Lambda_7\sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1\Lambda_2(\Lambda_3+\Lambda_4+\Lambda_5) + 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}},$

(i) $-2\sqrt{\Lambda_1\Lambda_2} \leq \Lambda_3 + \Lambda_4 + \Lambda_5 \leq 6\sqrt{\Lambda_1\Lambda_2},$

(ii) $\Lambda_3 + \Lambda_4 + \Lambda_5 > 6\sqrt{\Lambda_1\Lambda_2}$ and

$|\Lambda_6\sqrt{\Lambda_2} + \Lambda_7\sqrt{\Lambda_1}| \leq 2\sqrt{\Lambda_1\Lambda_2(\Lambda_3+\Lambda_4+\Lambda_5) - 2\Lambda_1\Lambda_2\sqrt{\Lambda_1\Lambda_2}},$

where $\Delta = 4(12\Lambda_1\Lambda_2 - 12\Lambda_6\Lambda_7 + (\Lambda_3+\Lambda_4+\Lambda_5)^2)^3 - (72\Lambda_1\Lambda_2(\Lambda_3+\Lambda_4+\Lambda_5) + 36\Lambda_6\Lambda_7(\Lambda_3+\Lambda_4+\Lambda_5) - 2(\Lambda_3+\Lambda_4+\Lambda_5)^3 - 108\Lambda_1\Lambda_7^2 - 108\Lambda_6^2\Lambda_2)^2$.

3 Vacuum stability of the general 2HDM potential

3.1 Sufficient conditions

In this section, we mainly give the vacuum stability conditions of the 2HDM potential (1) with explicit CP violation. We rewrite the quartic part of such a 2HDM potential as follow

$$\begin{aligned} V_4(\Phi_1,\Phi_2) =& \sum_{i,j,k,l=1}^{2} t_{ijkl}(\Phi_i^*\Phi_j)(\Phi_k^*\Phi_l) \\ =&\lambda_1(\Phi_1^*\Phi_1)^2 + \lambda_2(\Phi_2^*\Phi_2)^2 \\ &+ \lambda_3(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2) + \lambda_4(\Phi_1^*\Phi_2)(\Phi_2^*\Phi_1) \\ &+ \frac{\lambda_5}{2}(\Phi_1^*\Phi_2)^2 + \frac{\lambda_5^*}{2}(\Phi_2^*\Phi_1)^2 \\ &+ (\Phi_1^*\Phi_1)(\lambda_6\Phi_1^*\Phi_2 + \lambda_6^*\Phi_2^*\Phi_1) \\ &+ (\Phi_2^*\Phi_2)(\lambda_7\Phi_1^*\Phi_2 + \lambda_7^*\Phi_2^*\Phi_1). \end{aligned} \tag{11}$$

Let ϕ_i=$|\Phi_i| = \sqrt{\Phi_i^*\Phi_i}$, the modulus of Φ_i for $i = 1,2$. Then

$$\Phi_1^*\Phi_2 = \phi_1\phi_2\rho e^{i\theta} \text{ and } \Phi_2^*\Phi_1 = \phi_1\phi_2\rho e^{-i\theta},$$

here $i^2 = -1$ and $\rho \in [0,1]$ is the orbit space parameter [10,17,20]. Let

$$\lambda_5 = |\lambda_5|e^{i\varphi_5}, \lambda_6 = |\lambda_6|e^{i\varphi_6}, \lambda_7 = |\lambda_7|e^{i\varphi_7},$$

where φ_k is argument of the complex number λ_k $(k = 5,6,7)$. Then

$$\lambda_5^* = |\lambda_5|e^{-i\varphi_5}, \lambda_6^* = |\lambda_6|e^{-i\varphi_6}, \lambda_7^* = |\lambda_7|e^{-i\varphi_7}.$$

So, we have

$$\begin{aligned}
V_4(\Phi_1,\Phi_2) =&\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2 + \lambda_4\rho^2\phi_1^2\phi_2^2 \\
&+ \frac{|\lambda_5|}{2}(e^{i(\varphi_5+2\theta)} + e^{-i(\varphi_5+2\theta)})\phi_1^2\phi_2^2\rho^2 \\
&+ |\lambda_6|(e^{i(\varphi_6+\theta)} + e^{-i(\varphi_6+\theta)})\phi_1^3\phi_2\rho \\
&+ |\lambda_7|(e^{i(\varphi_7+\theta)} + e^{-i(\varphi_7+\theta)})\phi_1\phi_2^3\rho \\
=&\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2 + \lambda_4\rho^2\phi_1^2\phi_2^2 \\
&+ |\lambda_5|\phi_1^2\phi_2^2\rho^2\cos(\varphi_5 + 2\theta) \\
&+ 2|\lambda_6|\phi_1^3\phi_2\rho\cos(\varphi_6 + \theta) \\
&+ 2|\lambda_7|\phi_1\phi_2^3\rho\cos(\varphi_7 + \theta) \\
=&\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2 + \lambda_4\rho^2\phi_1^2\phi_2^2 \\
&+ |\lambda_5|\phi_1^2\phi_2^2\rho^2(\cos\varphi_5\cos 2\theta - \sin\varphi_5\sin 2\theta) \\
&+ 2|\lambda_6|\phi_1^3\phi_2\rho(\cos\varphi_6\cos\theta - \sin\varphi_6\sin\theta) \\
&+ 2|\lambda_7|\phi_1\phi_2^3\rho(\cos\varphi_7\cos\theta - \sin\varphi_7\sin\theta).
\end{aligned}$$

Obviously, $\mathbf{Re}\lambda_k = |\lambda_k|\cos\varphi_k, \mathbf{Im}\lambda_k = |\lambda_k|\sin\varphi_k, (k = 5,6,7)$. Then, noticing $\sin 2\theta = 2\sin\theta\cos\theta$, we have

$$\begin{aligned}
V_4(\Phi_1,\Phi_2) =&\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4\rho^2 + \mathbf{Re}\lambda_5\rho^2\cos 2\theta)\phi_1^2\phi_2^2 \\
&+ 2(\rho\mathbf{Re}\lambda_6\cos\theta)\phi_1^3\phi_2 + 2(\rho\mathbf{Re}\lambda_7\cos\theta)\phi_2^3\phi_1 \\
&- 2\rho\phi_1\phi_2(\mathbf{Im}\lambda_5\rho\phi_1\phi_2\sin\theta\cos\theta \\
&\qquad + \mathbf{Im}\lambda_6\phi_1^2\sin\theta + \mathbf{Im}\lambda_7\phi_2^2\sin\theta) \\
=&V_4'(\phi_1,\phi_2) + V_4''(\phi_1,\phi_2),
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
V_4'(\phi_1,\phi_2) =&\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4\rho^2 + \mathbf{Re}\lambda_5\rho^2\cos 2\theta)\phi_1^2\phi_2^2 \\
&+ 2(\rho\mathbf{Re}\lambda_6\cos\theta)\phi_1^3\phi_2 + 2(\rho\mathbf{Re}\lambda_7\cos\theta)\phi_2^3\phi_1, \\
V_4''(\phi_1,\phi_2) =& - 2(\rho\sin\theta)[(\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 \\
&\qquad + \mathbf{Im}\lambda_6\phi_1^2 + \mathbf{Im}\lambda_7\phi_2^2]\phi_1\phi_2.
\end{aligned} \tag{13}$$

Applying the co-positivity of a real tensor (9) with

$$\Lambda_i = \lambda_i \ (i = 1,2,3,4), \Lambda_k = \mathbf{Re}\lambda_k \ (k = 5,6,7)$$

to obtain that $\lambda_1 > 0, \lambda_2 > 0$,

$$V_4'(\phi_1, \phi_2) \geq 0 \text{ for all } \phi_1, \phi_2 \tag{14}$$

if and only if

(I) $\mathbf{Re}\lambda_6 = \mathbf{Re}\lambda_7 = 0, \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0,$
$\lambda_3 + \lambda_4 - |\mathbf{Re}\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0;$

(II) $\mathbf{Re}\lambda_6 \neq 0$ or $\mathbf{Re}\lambda_7 \neq 0, \Delta \geq 0,$
$\lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0, \lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0$
$|\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}},$
(i) $-2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2},$
(ii) $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2}$ and
$|\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}},$

where $\Delta = 4(12\lambda_1\lambda_2 - 12\mathbf{Re}\lambda_6 \cdot \mathbf{Re}\lambda_7 + (\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5)^2)^3 - (72\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 36\mathbf{Re}\lambda_6 \cdot \mathbf{Re}\lambda_7(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5)^3 - 108\lambda_1(\mathbf{Re}\lambda_7)^2 - 108(\mathbf{Re}\lambda_6)^2\lambda_2)^2$.

After making simple calculations ($\sin\theta \neq 0$), we have

$$V_4''(\phi_1, \phi_2) \geq 0 \text{ for all } \phi_1, \phi_2 \tag{15}$$

if and only if

$$\begin{cases} \mathbf{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 + \mathbf{Im}\lambda_7\phi_2^2 \geq 0, \ \sin\theta < 0, \\ \mathbf{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 + \mathbf{Im}\lambda_7\phi_2^2 \leq 0, \ \sin\theta > 0. \end{cases}$$

By the co-positivity of a real matrix (6) with

$$\mu_{11} = \mathbf{Im}\lambda_6, \mu_{12} = \frac{1}{2}(\rho\cos\theta)\mathbf{Im}\lambda_5, \mu_{22} = \mathbf{Im}\lambda_7,$$

we obtain that

$$\mathbf{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 + \mathbf{Im}\lambda_7\phi_2^2 \geq 0 \text{ for all } \phi_1, \phi_2$$

if and only if

$$\mathbf{Im}\lambda_6 \geq 0, \mathbf{Im}\lambda_7 \geq 0, (\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Case 1: $\mathbf{Im}\lambda_5 \geq 0$.

The function $g(\rho\cos\theta) = (\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7}$ reaches its minimum at $\rho\cos\theta = -1$, so for all $\rho\cos\theta \in [-1, 1]$, we have

$$(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0 \Leftrightarrow -\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Case 2: $\mathbf{Im}\lambda_5 < 0$.

The function $g(\rho\cos\theta) = (\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7}$ reaches its minimum at $\rho\cos\theta = 1$, so for all $\rho\cos\theta \in [-1, 1]$, we have

$$(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0 \Leftrightarrow \mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Therefore, for any real number $\mathbf{Im}\lambda_5$, we have

$$(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0 \Leftrightarrow -|\mathbf{Im}\lambda_5| + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Similarly, we also have

$$\mathbf{Im}\lambda_6\phi_1^2 + (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 + \mathbf{Im}\lambda_7\phi_2^2 \leq 0 \text{ for all } \phi_1, \phi_2$$

if and only if

$$-\mathbf{Im}\lambda_6\phi_1^2 - (\rho\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 - \mathbf{Im}\lambda_7\phi_2^2 \geq 0 \text{ for all } \phi_1, \phi_2$$

which is equivalent to the co-positivity of 2×2 matrix $M = (\mu_{ij})$ with its entries,

$$\mu_{11} = -\mathbf{Im}\lambda_6, \mu_{12} = -\frac{1}{2}(\rho\cos\theta)\mathbf{Im}\lambda_5, \mu_{22} = -\mathbf{Im}\lambda_7.$$

This means that

$$-\mathbf{Im}\lambda_6 \geq 0, -\mathbf{Im}\lambda_7 \geq 0, -(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0,$$

that is,

$$\mathbf{Im}\lambda_6 \leq 0, \mathbf{Im}\lambda_7 \leq 0, -(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

Similarly, the function

$$f(\rho\cos\theta) = -(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7}$$

reaches its minimum at $\rho\cos\theta = 1$ ($\mathbf{Im}\lambda_5 \geq 0$) or -1 ($\mathbf{Im}\lambda_5 < 0$), and hence, for any real number $\mathbf{Im}\lambda_5$,

$$-(\rho\cos\theta)\mathbf{Im}\lambda_5 + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0 \Leftrightarrow -|\mathbf{Im}\lambda_5| + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

So, we get the conclusion that

$$V_4''(\phi_1, \phi_2) \geq 0 \text{ for all } \phi_1, \phi_2$$

if and only if

$$\textbf{(III)} \qquad \mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7 \geq 0, \ -|\mathbf{Im}\lambda_5| + 2\sqrt{\mathbf{Im}\lambda_6 \cdot \mathbf{Im}\lambda_7} \geq 0.$$

In summary, we prove the analytic conditions (**I**), (**II**) and (**III**) assure the vacuum stability of the 2HDM potential with explicit CP violation. At the same time, we also obtain the semi-positive definiteness of a 4th-order 2-dimensional complex tensor $\mathcal{T} = (t_{ijkl})$ definited by the Eq. (5).

In term of Eq. (8), we also may obtain a stronger sufficient condition. That is, $V_4(\Phi_1, \Phi_2) \geq 0$ for all Φ_1, Φ_2 if for all $\rho \in [0,1]$ and all $\theta \in [0, 2\pi]$,

$$\begin{aligned}
&\beta(\theta) = 2\rho|\lambda_6|\cos(\varphi_6 + \theta) + 4\sqrt[4]{\lambda_1^3\lambda_2} \geq 0,\\
&\gamma(\theta) = 2\rho|\lambda_7|\cos(\varphi_7 + \theta) + 4\sqrt[4]{\lambda_1\lambda_2^3} \geq 0,\\
&(\lambda_3 + \lambda_4\rho^2 + |\lambda_5|\rho^2\cos(\varphi_5 + 2\theta)) - 6\sqrt{\lambda_1\lambda_2} + 2\sqrt{\beta(\theta)\cdot\gamma(\theta)} \geq 0.
\end{aligned}$$

It is obvious that the above inequality system comes true if

$$
\textbf{(IV)}\qquad
\begin{aligned}
&\beta = 2\sqrt[4]{\lambda_1^3\lambda_2} - |\lambda_6| \geq 0, \gamma = 2\sqrt[4]{\lambda_1\lambda_2^3} - |\lambda_7| \geq 0,\\
&\lambda_3 - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq 0,\\
&\lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq 0.
\end{aligned}
$$

Similarly, $V_4'(\phi_1, \phi_2) \geq 0$ for all ϕ_1, ϕ_2 if

$$
\textbf{(IV}'\textbf{)}\qquad
\begin{aligned}
&\beta' = 2\sqrt[4]{\lambda_1^3\lambda_2} - |\mathbf{Re}\lambda_6| \geq 0, \gamma' = 2\sqrt[4]{\lambda_1\lambda_2^3} - |\mathbf{Re}\lambda_7| \geq 0,\\
&\lambda_3 - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta'\gamma'} \geq 0,\\
&\lambda_3 + \lambda_4 - |\mathbf{Re}\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta'\gamma'} \geq 0.
\end{aligned}
$$

Remark 3.1. Four analytical sufficient conditions are following:

(1) $\mathbf{Re}\lambda_6 = \mathbf{Re}\lambda_7 = 0$, the conditions (**I**) and (**III**);

(2) $\mathbf{Re}\lambda_6 \neq 0$ or $\mathbf{Re}\lambda_7 \neq 0$, the conditions (**II**) and (**III**);

(3) the conditions (**IV**$'$) and (**III**);

(4) the conditions (**IV**).

These analytical sufficient conditions are obtained from the real and imaginary parts of complex numbers, not only dependent of the norm. So they are different from the ones of Bahl et.al. [22]. For example,

$$\lambda_1 = \lambda_2 = 1, \lambda_3 = 6, \lambda_4 = 2, \lambda_5 = -1 - i, \lambda_6 = \lambda_7 = -1 - \sqrt{3}i.$$

Obviously, $|\lambda_6| = |\lambda_7| = 2, \beta = \gamma = 0, \lambda_3 - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} = 0,$

$$\lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} = 2 - \sqrt{2} > 0.$$

That is, these parameters meet the condition (**IV**), which means $V_4(\Phi_1, \Phi_2) \geq 0$. It is obvious that the conditions (**II**), (**IV**$'$) and (**III**) are fulfiled also. However, they can't satisfy the condition Eq. (5.20) of Bahl et.al. [22], i.e.

$$
\begin{aligned}
&3\sqrt{\lambda_1\lambda_2} - (\lambda_3 + |\lambda_4| + |\lambda_5|) = 3 - (6 + 2 + \sqrt{2}) < 0,\\
&\sqrt{\lambda_1\lambda_2} + \lambda_3 - (|\lambda_4| + |\lambda_5| + 4\left|\lambda_6\sqrt[4]{\frac{\lambda_2}{\lambda_1}}\right|) = 1 + 6 - (2 + \sqrt{2} + 8) < 0,\\
&\sqrt{\lambda_1\lambda_2} + \lambda_3 - (|\lambda_4| + |\lambda_5| + 4\left|\lambda_7\sqrt[4]{\frac{\lambda_1}{\lambda_2}}\right|) = 1 + 6 - (2 + \sqrt{2} + 8) < 0.
\end{aligned}
$$

Also see the example in the introduce.

3.2 Sufficient and necessary conditions

In this subsection, $V_4(\Phi_1, \Phi_2)$ is rewritten as follows $(\lambda_1 > 0, \lambda_2 > 0)$,

$$\begin{aligned}
&V_4(\Phi_1, \Phi_2) = A\rho^2 + B\rho + C = f(\rho),\\
&A = a\phi_1^2\phi_2^2,\ B = b\phi_1\phi_2,\ C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2,\\
&a = \lambda_4 - \mathbf{Re}\lambda_5 + 2(\mathbf{Re}\lambda_5\cos\theta - \mathbf{Im}\lambda_5\sin\theta)\cos\theta\\
&\quad = \lambda_4 + |\lambda_5|\cos(\varphi_5 + 2\theta),\\
&b = 2(\mathbf{Re}\lambda_6\phi_1^2 + \mathbf{Re}\lambda_7\phi_2^2)\cos\theta - 2(\mathbf{Im}\lambda_6\phi_1^2 + \mathbf{Im}\lambda_7\phi_2^2)\sin\theta\\
&\quad = 2|\lambda_6|\phi_1^2\cos(\varphi_6 + \theta) + 2|\lambda_7|\phi_2^2\cos(\varphi_7 + \theta).
\end{aligned} \tag{16}$$

The quadratic function $f(\rho)$ is non-negative about a variable $\rho \in [0, 1]$ if and only if its minimum is non-negative in the interval $[0, 1]$, and so, its function value is non-negative at the boundary points $\rho = 0, 1$ and the unique minimum point $\rho_0 = -\frac{B}{2A} \in [0, 1](A > 0)$. That is,

$$f(\rho) \geq 0 \Leftrightarrow \begin{cases} f(-\frac{B}{2A}) = \dfrac{4AC - B^2}{4A} \geq 0, -\frac{B}{2A} \in [0, 1],\\ f(0) \geq 0,\\ f(1) \geq 0. \end{cases}$$

The graph-like of $f(\rho)$ is as shown below:

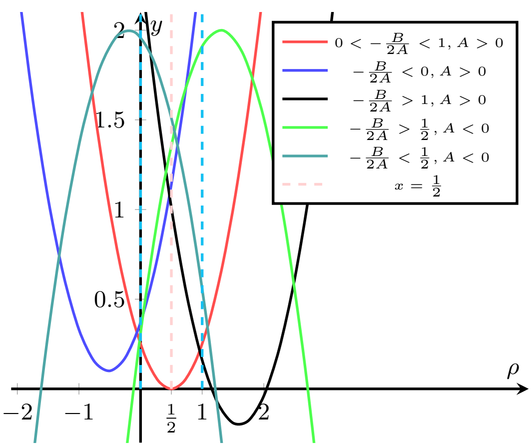

Figure 1: Graph-like of $f(\rho)$

Proposition 1. $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if

$$\begin{cases} 4AC - B^2 \geq 0, \quad -2A \leq B \leq 0;\\ C \geq 0;\\ A + B + C \geq 0. \end{cases}$$

It is obvious that if $\lambda_6 = \lambda_7 = 0$, then $B = 0$, the symmetry axis $-\frac{B}{2A} = 0$, and hence, $V_4(\Phi_1, \Phi_2) \geq 0$ if and only if

$$C \geq 0 \text{ and } A + C \geq 0.$$

For the 2HDM with $\mathbb{Z}_2$ symmetry [12], the quartic part of the general 2HDM scalar potential is

$$V_4^{\mathbb{Z}_2}(\Phi_1,\Phi_2) = \lambda_1(\Phi_1^*\Phi_1)^2 + \lambda_2(\Phi_2^*\Phi_2)^2 + \lambda_3(\Phi_1^*\Phi_1)(\Phi_2^*\Phi_2) + \lambda_4(\Phi_1^*\Phi_2)(\Phi_2^*\Phi_1) + \frac{\lambda_5}{2}(\Phi_1^*\Phi_2)^2 + \frac{\lambda_5^*}{2}(\Phi_2^*\Phi_1)^2. \tag{17}$$

Therefore, $V_4^{\mathbb{Z}_2}(\Phi_1,\Phi_2) \geq 0$ if and only if

$$\begin{cases} C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + \lambda_3\phi_1^2\phi_2^2 \geq 0 \\ A + C = \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 + |\lambda_5|\cos(\varphi_5 + 2\theta))\phi_1^2\phi_2^2 \geq 0. \end{cases}$$

It is clear that $C \geq 0 \Leftrightarrow \lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0$,

$$\begin{aligned} A + C \geq 0 \Leftrightarrow & \lambda_3 + \lambda_4 + |\lambda_5|\cos(\varphi_5 + 2\theta) + 2\sqrt{\lambda_1\lambda_2} \geq 0, \forall\theta \in [0, 2\pi] \\ \Leftrightarrow & \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0. \end{aligned}$$

Corollary 2. $V_4^{\mathbb{Z}_2}(\Phi_1,\Phi_2) \geq 0$ if and only if $\lambda_1 \geq 0, \lambda_2 \geq 0$,

(**V**) $$\lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0, \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0.$$

This condition (**V**) is well-known for the inert doublet model [4,10,11,18,27].

3.3 Necessary conditions

In this subsection, $V_4(\Phi_1,\Phi_2)$ is rewritten as follows ($\lambda_1 > 0, \lambda_2 > 0$),

$$\begin{aligned} V_4(\Phi_1,\Phi_2) &= f(\rho) = A\rho^2 + B\rho + C, \\ f(1) &= A + B + C \\ &= \lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 + 2\mathbf{Re}\lambda_5\cos^2\theta)\phi_1^2\phi_2^2 \\ &\quad + 2(\mathbf{Re}\lambda_6\cos\theta)\phi_1^3\phi_2 + 2(\mathbf{Re}\lambda_7\cos\theta)\phi_2^3\phi_1 \\ &\quad - 2(\sin\theta)[(\cos\theta)\mathbf{Im}\lambda_5\phi_1\phi_2 \\ &\qquad + \mathbf{Im}\lambda_6\phi_1^2 + \mathbf{Im}\lambda_7\phi_2^2]\phi_1\phi_2. \end{aligned} \tag{18}$$

Obviously, we have

$$f(\rho) \geq 0, \text{ for all } \rho \in [0,1] \Rightarrow f(0) \geq 0, f(1) \geq 0.$$

Then $V_4(\Phi_1,\Phi_2) \geq 0$ implies that $f(0) = C \geq 0$, which is equvalent to

(**VI**) $$\lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0.$$

This is a necessary condition of the vacuum stability of the general 2HDM potential. Clearly, the other necessary condition is some conditions such that $f(1) = A+B+C \geq 0$. By Eq. (18), it is known that $A+B+C$ may be regarded as a quartic form with two parameters $t = \sin\theta$ and $s = \cos\theta$ with $s^2+t^2 = 1$. So, when $s = \sin\theta = 0$ and $t = \cos\theta = \pm 1$, the inequality

$$\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5)\phi_1^2\phi_2^2 \pm 2\mathbf{Re}\lambda_6\phi_1^3\phi_2 \pm 2\mathbf{Re}\lambda_7\phi_2^3\phi_1 \geq 0$$

if and only if (using Eq.(7))

(1) $\Delta \leq 0$, $\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1} > 0$;

(2) $\mathbf{Re}\lambda_6 \geq 0$, $\mathbf{Re}\lambda_7 \geq 0$, $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0$;

(3) $\Delta \geq 0$,

$|\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}$,

(i) $-2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2}$,

(ii) $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2}$,

$\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1} \geq -2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}$.

and

(1′) $\Delta \leq 0$, $-\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1} > 0$;

(2′) $-\mathbf{Re}\lambda_6 \geq 0$, $-\mathbf{Re}\lambda_7 \geq 0$, $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0$;

(3′) $\Delta \geq 0$,

$|-\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}$,

(i') $-2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2}$,

(ii') $\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2}$,

$-\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1} \geq -2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}$.

Which is equivalent to

$$\begin{aligned}
&\mathbf{Re}\lambda_6 = \mathbf{Re}\lambda_7 = 0, \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0;\\
&\mathbf{Re}\lambda_6 \neq 0 \text{ or } \mathbf{Re}\lambda_7 \neq 0, \Delta \geq 0,\\
&|\mathbf{Re}\lambda_6\sqrt{\lambda_2} - \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}\\
&\quad (a) - 2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2},\\
&\quad (b) \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2},\\
&|\mathbf{Re}\lambda_6\sqrt{\lambda_2} + \mathbf{Re}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}.
\end{aligned}$$

So, a necessary condition of $V_4(\Phi_1, \Phi_2) \geq 0$ is

$$\textbf{(VII)} \qquad \lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0.$$

Similarly, if $t = \cos\theta = 0$ and $s = \sin\theta = \pm 1$, the inequality

$$\lambda_1\phi_1^4 + \lambda_2\phi_2^4 + (\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5)\phi_1^2\phi_2^2 \pm 2\mathbf{Im}\lambda_6\phi_1^3\phi_2 \pm 2\mathbf{Im}\lambda_7\phi_2^3\phi_1 \geq 0$$

is equivalent to

$$\begin{aligned}
&\mathbf{Im}\lambda_6 = \mathbf{Im}\lambda_7 = 0, \lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} \geq 0;\\
&\mathbf{Im}\lambda_6 \neq 0 \text{ or } \mathbf{Im}\lambda_7 \neq 0, \Delta' \geq 0,\\
&|\mathbf{Im}\lambda_6\sqrt{\lambda_2} - \mathbf{Im}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5) + 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}}\\
&\quad (a') - 2\sqrt{\lambda_1\lambda_2} \leq \lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 \leq 6\sqrt{\lambda_1\lambda_2},\\
&\quad (b') \lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 > 6\sqrt{\lambda_1\lambda_2},\\
&|\mathbf{Im}\lambda_6\sqrt{\lambda_2} + \mathbf{Im}\lambda_7\sqrt{\lambda_1}| \leq 2\sqrt{\lambda_1\lambda_2(\lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5) - 2\lambda_1\lambda_2\sqrt{\lambda_1\lambda_2}},
\end{aligned}$$

where $\Delta' = 4(12\lambda_1\lambda_2 - 12\mathbf{Im}\lambda_6\cdot\mathbf{Im}\lambda_7 + (\lambda_3+\lambda_4-\mathbf{Re}\lambda_5)^2)^3 - (72\lambda_1\lambda_2(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5) + 36\mathbf{Im}\lambda_6\cdot\mathbf{Im}\lambda_7(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5) - 2(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5)^3 - 108\lambda_1(\mathbf{Im}\lambda_7)^2 - 108(\mathbf{Im}\lambda_6)^2\lambda_2)^2$. So, a necessary condition of $V_4(\Phi_1,\Phi_2)\geq 0$ is

$$(\mathbf{VIII}) \qquad \lambda_3+\lambda_4-\mathbf{Re}\lambda_5+2\sqrt{\lambda_1\lambda_2}\geq 0.$$

Applying the Corollary 3.1 of Song and Qi [33] to $V_4(\Phi_1,\Phi_2)$,

$$\begin{aligned}V_4(\Phi_1,\Phi_2) =&\lambda_1\phi_1^4+\lambda_2\phi_2^4+[\lambda_3+\lambda_4\rho^2+|\lambda_5|\rho^2\cos(\varphi_5+2\theta)]\phi_1^2\phi_2^2\\&+2|\lambda_6|\phi_1^3\phi_2\rho\cos(\varphi_6+\theta)+2|\lambda_7|\phi_1\phi_2^3\rho\cos(\varphi_7+\theta),\end{aligned}$$

we obtain that $V_4(\Phi_1,\Phi_2)>0$ implies that for all $\rho\in[0,1]$ and all $\theta\in[0,2\pi]$,

$$\begin{aligned}0<&2\left(\frac{1}{4}\times 2|\lambda_6|\rho\cos(\varphi_6+\theta)\right)\sqrt{\lambda_2}+2\left(\frac{1}{4}\times 2|\lambda_7|\rho\cos(\varphi_7+\theta)\right)\sqrt{\lambda_1}\\&+\left(3\times\frac{1}{6}(\lambda_3+\lambda_4\rho^2+|\lambda_5|\rho^2\cos(\varphi_5+2\theta))+\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2}\\=&|\lambda_6|\sqrt{\lambda_2}\rho\cos(\varphi_6+\theta)+|\lambda_7|\sqrt{\lambda_1}\rho\cos(\varphi_7+\theta)\\&+\frac{1}{2}\left(\lambda_3+\lambda_4\rho^2+|\lambda_5|\rho^2\cos(\varphi_5+2\theta)+2\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2}\\=&\mathbf{Re}\lambda_6\sqrt{\lambda_2}\rho\cos\theta+\mathbf{Re}\lambda_7\sqrt{\lambda_2}\rho\cos\theta\\&+\frac{1}{2}\left(\lambda_3+\lambda_4\rho^2+\mathbf{Re}\lambda_5\rho^2\cos 2\theta+2\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2}\\&-\mathbf{Im}\lambda_6\sqrt{\lambda_2}\rho\sin\theta-\mathbf{Im}\lambda_7\sqrt{\lambda_1}\rho\sin\theta-\frac{1}{2}\mathbf{Im}\lambda_5\sqrt[4]{\lambda_1\lambda_2}\rho^2\sin 2\theta,\end{aligned}$$

and then, $\rho=1$ and $\theta=0$ or π or $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, the above inequality must holds also. That is,

$$\begin{aligned}&\left(\lambda_3+\lambda_4+\mathbf{Re}\lambda_5+2\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2}\pm 2\left(\mathbf{Re}\lambda_6\sqrt{\lambda_2}+\mathbf{Re}\lambda_7\sqrt{\lambda_1}\right)>0\\&\left(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5+2\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2}\mp 2\left(\mathbf{Im}\lambda_6\sqrt{\lambda_2}+\mathbf{Im}\lambda_7\sqrt{\lambda_1}\right)>0.\end{aligned}$$

or equivalently,

$$(\mathbf{IX}) \qquad \begin{aligned}&\left(\lambda_3+\lambda_4+\mathbf{Re}\lambda_5+2\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2}\\&\qquad -2\left|\mathbf{Re}\lambda_6\sqrt{\lambda_2}+\mathbf{Re}\lambda_7\sqrt{\lambda_1}\right|>0\\&\left(\lambda_3+\lambda_4-\mathbf{Re}\lambda_5+2\sqrt{\lambda_1\lambda_2}\right)\sqrt[4]{\lambda_1\lambda_2}\\&\qquad -2\left|\mathbf{Im}\lambda_6\sqrt{\lambda_2}+\mathbf{Im}\lambda_7\sqrt{\lambda_1}\right|>0.\end{aligned}$$

Clearly, the condition (**VI**) (strict inequality) is obtained if $\rho=0$.

In summary, the conditions (**VI**), (**VII**), (**VIII**) and (**IX**) are the necessary conditions of the vacuum stability of the general 2HDM potential.

Remark 3.2. The conditions (**I**) and (**II**) obviuosly meet (**VI**)-(**VIII**) (necessary). For (**IV**) and (**IV′**),

$$\beta = 2\sqrt[4]{\lambda_1^3\lambda_2} - |\lambda_6| \geq 0, \gamma = 2\sqrt[4]{\lambda_1\lambda_2^3} - |\lambda_7| \geq 0,$$

$$\begin{aligned}\lambda_3 + 2\sqrt{\lambda_1\lambda_2} &= \lambda_3 - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{(2\sqrt[4]{\lambda_1^3\lambda_2})(2\sqrt[4]{\lambda_1\lambda_2^3})} \\ &\geq \lambda_3 - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq 0,\end{aligned}$$

i.e., $\lambda_3 + 2\sqrt{\lambda_1\lambda_2} \geq 0$. So, the condition (VI) holds. Similarly,

$$\begin{aligned}&\lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{(2\sqrt[4]{\lambda_1^3\lambda_2})(2\sqrt[4]{\lambda_1\lambda_2^3})} \\ &\geq \lambda_3 + \lambda_4 - |\lambda_5| - 6\sqrt{\lambda_1\lambda_2} + 4\sqrt{\beta\gamma} \geq 0,\end{aligned}$$

i.e., $\lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0$. It follows from the inequalities $-|\lambda_5| \leq \mathbf{Re}\lambda_5 \leq |\lambda_5|$ that

$$\begin{aligned}\lambda_3 + \lambda_4 + \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} &\geq \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0, \\ \lambda_3 + \lambda_4 - \mathbf{Re}\lambda_5 + 2\sqrt{\lambda_1\lambda_2} &\geq \lambda_3 + \lambda_4 - |\lambda_5| + 2\sqrt{\lambda_1\lambda_2} \geq 0.\end{aligned}$$

So, the conditions (**VII**) and (**VIII**) hold. Similarly, the condition (**IV′**) meets (**VI**)-(**VIII**) also. The condition (**IX**) is necessary condition of "strict inequality", $V_4(\Phi_1, \Phi_2) > 0$.

4 Conclusions

By means of the co-positive conditions of a 4th-order symmetry tensor, several analytical sufficient conditions and necessary conditions are established for the vacuum stability of the general 2HDM potential, respectively. That is,

$$\text{Four sufficient conditions: } \begin{cases}(1)\ (\mathbf{I}) \text{ and } (\mathbf{III}); \\ (2)\ (\mathbf{II}) \text{ and } (\mathbf{III}); \\ (3)\ (\mathbf{IV'}) \text{ and } (\mathbf{III}); \\ (4)\ (\mathbf{IV}).\end{cases}$$

Four necessary conditions: (**VI**), (**VII**), (**VIII**) and (**IX**) .

A sufficient and necessary condition is qualitatively showed for the vacuum stability of the general 2HDM potential, and then, applying it to derive the analytical necessary conditions for the vacuum stability of the general 2HDM potential. The vacuum stability condition (**V**) of the $\mathbb{Z}_2$ symmetry 2HDM potential is a special case.

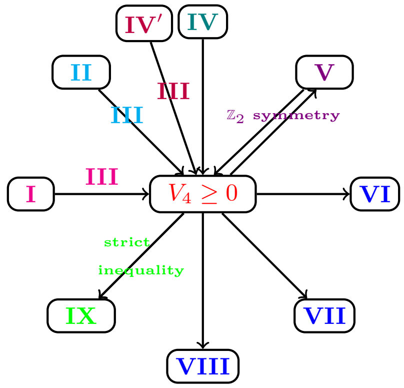

Figure 2: Analytical conditions and the vacuum stability of the general 2HDM potential ("→" stand for "imply")

Competing interest

The authors declare that they have no known competing financial interests or personal rela-tionships that could have appeared to influence the work reported in this paper.

Authors' contributions

Yisheng Song is the sole author.

Availability of data and materials

This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and there are no external data associated with the manuscript].

Funding

This work was supported by the National Natural Science Foundation of P.R. China (Grant No.12171064), by The team project of innovation leading talent in chongqing (No.CQYC20210309536) and by the Foundation of Chongqing Normal university (20XLB009).

Acknowledgements

The authors would like to express their sincere thanks to the Editors and the anonymous referees for their constructive comments and valuable suggestions.

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